Bisimulations for Probabilistic and Quantum Processes

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Abstract

Bisimulation is a fundamental concept in the classical concurrency theory for comparing the behaviour of nondeterministic processes. It admits elegant characterisations from various perspectives such as fixed point theory, modal logics, game theory, coalgebras etc. In this paper, we review some key ideas used in the formulations and characterisations of reasonable notions of bisimulations for both probabilistic and quantum processes. To some extent the transition from probabilistic to quantum concurrency theory is smooth and natural. However, new ideas need also to be introduced. We have not yet reached the stage of formally verifying quantum communication protocols and quantum algorithms using bisimulations implemented by automatic tools. We discuss some recent efforts in this direction.

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1 Introduction

Bisimulation [39, 37] is a fundamental concept in the classical concurrency theory as it admits beautiful characterisations in terms of fixed points, modal logics, co-algebras, pseudometrics, games, decision algorithms, etc. Its generalisation in the probabilistic setting is initiated by Larsen and Skou in [36] and has subsequently been widely investigated in probabilistic concurrency theory. One of the main contributions of [36] is the introduction of a lifting operation that converts a relation between states to a relation between distributions over states. Later on, the lifting operation is shown to be closely related to some prominent concepts in mathematics such as the Kantorovich metric [33, 45] and the maximum network flow problem [1]; the latter is crucial for designing algorithms to check if two states are bisimilar.

The probabilistic bisimulation nicely defined in [36] has natural characterisations by probabilistic extensions of Hennessy-Milner logic [28]; see e.g. [36, 14, 15, 40, 10, 30, 25, 12, 4]. Most characterisations employ some modalities indexed with numbers. A typical modal formula, dated back to [36], is $\langle a \rangle_p \phi$, where p is a probability value. A state s satisfies this

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formula if the probability that s can make an a-labelled transition to the set of states satisfying ϕ exceeds p. In [44] van Breugel et al. generalise the characterisation of [36] to labelled Markov processes, i.e. reactive probabilistic processes [36, 46] with continuous state spaces, and surprisingly, without using any modality indexed with numbers. Usually, the simpler the logical characterisation, the more difficult its completeness proof, since constructing distinguishing formulae for non-bisimilar states with fewer modalities is more challenging. Van Breugel et al. prove such an elegant result by using some advanced machinery such as the Lawson topology on probabilistic powerdomains [31] and Banach algebras. However, if we confine ourselves to discrete rather than continuous state spaces, as in e.g. [36], the characterisation result given in [44] has a very elementary proof [7].

Since probabilistic behaviour is prevalent in quantum computation, it is natural to investigate how a quantum concurrency theory can be built upon the probabilistic concurrency theory. Notice that the operational semantics of many quantum systems can be defined in terms of probabilistic labelled transition systems, which allows us to define quantum bisimulations in a very intuitive way by extending probabilstic bisimulations with a requirement on demanding equal environments when comparing two quantum processes. However, to check quantum bisimulations, we need to appeal to the instantiation of quantum variables by quantum systems. What's worse, to check whether or not two quantum processes are bisimilar, we need to consider arbitrarily chosen quantum states, which appears infeasible in practice because quantum states constitute a continuum. Fortunately, it is possible to overcome this difficulty by introducing a symbolic semantics and its associated symbolic quantum bisimulations [18] that are equivalent to the usual concrete bisimulations. This opens the door to design effective algorithms to check quantum bisimulations.

A distinctive feature of quantum computation is entailed by the no-cloning theorem in quantum mechanics. Namely, quantum resources are linear from a type-theoretic point of view. It is then particularly meaningful to study *linear contextual equivalence*, which is a special form of contextual equivalence as the behaviours of programs are observed by executing them only once. In [8], it is shown that for higher-order quantum programs, linear contextual equivalence can be precisely captured by a distribution-based bisimilarity, which is weaker than the usual state-based bisimilarity. Of course, distribution-based bisimulations can also be defined for probabilistic processes, but in the quantum setting they become a more important coinductive proof technique.

The rest of the paper is structured as follows. In Section 2, we review the formal model of probabilistic labelled transition systems, the lifting operation, some of its equivalent formulations, state-based and distribution-based bisimulations. In Section 3 we introduce a quantum process algebra, discuss state-based and distribution-based quantum bisimulations, and symbolic bisimulations. Finally, we conclude in Section 4.

2 Probabilistic Bisimulation

In this section, we introduce the model of probabilistic labelled transition systems, the key concept of lifting operation, the state-based and distribution-based bisimulations.

2.1 Probabilistic Labelled Transition Systems

Let S be a countable set. A *(discrete) probability (sub)distribution* over set S is a function $\Delta : S \to [0,1]$ with *size* $|\Delta| = \sum_{s\in S} \Delta(s) \leq 1$. It is a *(full) distribution* if $|\Delta| = 1$. Its *support*, written $\lceil\Delta\rceil$, is the set $\{s \in S \mid \Delta(s) > 0\}$. Let $\mathcal{D}_{sub}(S)$ and $\mathcal{D}(S)$ denote the set of all subdistributions and distributions over S, respectively. We use ε to stand for the empty

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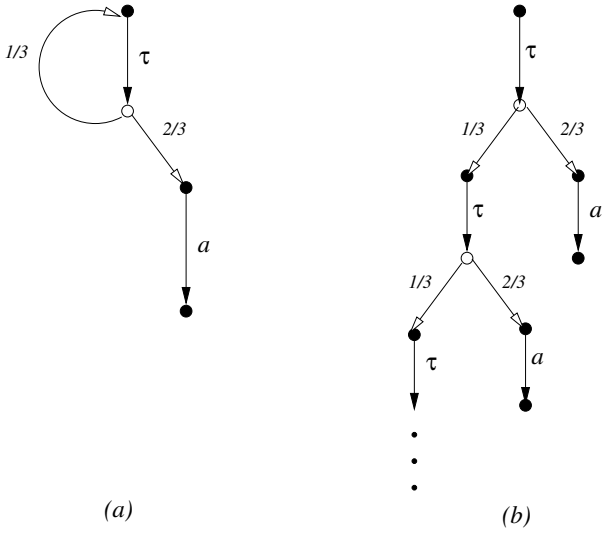

Figure 1 Example pLTSs

subdistribution, that is $\varepsilon(s) = 0$ for any $s \in S$. We write $\overline{s}$ for the point distribution for state s, satisfying $\overline{s}(t) = 1$ if $t = s$, and 0 otherwise. If $p_i \geq 0$ and Δ_i is a distribution for each i in some finite index set I, then $\sum_{i\in I} p_i \cdot \Delta_i$ is given by

$$(\sum_{i\in I} p_i \cdot \Delta_i)(s) \quad = \quad \sum_{i\in I} p_i \cdot \Delta_i(s) \ .$$

If $\sum_{i\in I} p_i = 1$ then this is easily seen to be a distribution in $\mathcal{D}(S)$.

▶ **Definition 1.** A *probabilistic labelled transition system* (pLTS) is defined as a triple $\langle S, A, \rightarrow\rangle$, where S is a set of states, A is a set of actions, and the transition relation $\rightarrow$ is a subset of $S \times A \times \mathcal{D}(S)$.

A non-probabilistic labelled transition system (LTS) may be viewed as a degenerate pLTS — one in which only point distributions are used. We often write $s \xrightarrow{\alpha} \Delta$ in place of $(s, \alpha, \Delta) \in \rightarrow$.

In order to visualise pLTSs, we often draw them as directed graphs. Given that in a pLTS transitions go from states to distributions, we need to introduce additional edges to connect distributions back to states, thereby obtaining a bipartite graph. States are therefore represented by nodes of the form • and distributions by nodes of the form ∘. For any state s and distribution Δ with $s \xrightarrow{\alpha} \Delta$ we draw an edge from s to Δ, labelled with α. Consequently, the edges leaving a •-node are all labelled with actions from A. For any distribution Δ and state s in $\lceil\Delta\rceil$, the support of Δ, we draw an edge from Δ to s, labelled with $\Delta(s)$. Consequently, the edges leaving a ∘-node are labelled with positive real numbers that sum to 1. Sometimes we partially unfold this graph by drawing the same nodes multiple times; in doing so, all outgoing edges of a given instance of a node are always drawn, but not necessarily all incoming edges. Edges labelled by probability 1 occur so frequently that it makes sense to omit them, together with the associated nodes ∘ representing point distributions.

Two example pLTSs are described this way in Figure 1, where diagram (b) depicts the initial part of the pLTS obtained by unfolding the one in diagram (a).

For each state s, the outgoing transition $s \xrightarrow{\alpha} \Delta$ represents the nondeterministic alternatives available in the state s. The nondeterministic choices provided by s are supposed

to be resolved by the environment, which is often formalised by a *scheduler* or an *adversary*. On the other hand, the probabilistic choices in the underlying distribution Δ are made by the system itself. Therefore, for each state s, the environment chooses some outgoing transition $s \xrightarrow{\alpha} \Delta$. Then the action α is performed, the system resolves the probabilistic choice, and subsequently with probability $\Delta(s')$ the system reaches state s'.

If we impose the constraint that for any state s and action α at most one outgoing transition from s is labelled α, then we obtain the special class of pLTSs called *reactive* (or *deterministic*) pLTSs that are the probabilistic counterpart to deterministic LTSs. Formally, a pLTS is reactive if for each $s \in S, \alpha \in A$ we have that $s \xrightarrow{\alpha} \Delta$ and $s \xrightarrow{\alpha} \Delta'$ imply $\Delta = \Delta'$.

2.2 Lifting Relations

In the probabilistic setting, formal systems are usually modelled as distributions over states. To compare two systems involves the comparison of two distributions. So we need a way of lifting relations on states to relations on distributions. This is used, for example, to define a notion of probabilistic bisimulation as we shall see soon. A few approaches of lifting relations have appeared in the literature. We will take the one from [11], and show its coincidence with two other approaches.

▶ **Definition 2.** Given two sets S and T and a binary relation $\mathcal{R} \subseteq S \times T$, the lifted relation $\mathcal{R}^\dagger \subseteq \mathcal{D}(S) \times \mathcal{D}(T)$ is the smallest relation that satisfies:

(1) $s \mathcal{R} t$ implies $\overline{s} \mathcal{R}^\dagger \overline{t}$

(2) (Linearity) $\Delta_i \mathcal{R}^\dagger \Theta_i$ for all $i \in I$ implies $(\sum_{i\in I} p_i \cdot \Delta_i) \mathcal{R}^\dagger (\sum_{i\in I} p_i \cdot \Theta_i)$, where I is a finite index set and $\sum_{i\in I} p_i = 1$.

There are alternative presentations of Definition 2. One example is given below.

▶ Proposition 3. Let Δ and Θ be two distributions over S and T, respectively, and $\mathcal{R} \subseteq S \times T$. Then $\Delta \mathcal{R}^\dagger \Theta$ if and only if there are two collections of states, $\{s_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$, and a collection of probabilities $\{p_i\}_{i\in I}$, for some finite index set I, such that $\sum_{i\in I} p_i = 1$ and Δ, Θ can be decomposed as follows:

(1) $\Delta = \sum_{i\in I} p_i \cdot \overline{s_i}$

(2) $\Theta = \sum_{i\in I} p_i \cdot \overline{t_i}$

(3) For each $i \in I$ we have $s_i \mathcal{R} t_i$.

From Definition 2, the next two propositions follow. In fact, they are sometimes used in the literature as definitions of lifting relations instead of being properties (see e.g. [43, 36, 13, 41]).

▶ Proposition 4. **(1)** Let Δ and Θ be distributions over S and T, respectively. Then $\Delta \mathcal{R}^\dagger \Theta$ if and only if there is a probability distribution on $S \times T$, with support a subset of $\mathcal{R}$, such that Δ and Θ are its marginal distributions. In other words, there exists a weight function $w : S \times T \to [0, 1]$ such that

a. $\forall s \in S : \sum_{t\in T} w(s, t) = \Delta(s)$

b. $\forall t \in T : \sum_{s\in S} w(s, t) = \Theta(t)$

c. $\forall (s, t) \in S \times T : w(s, t) > 0 \Rightarrow s \mathcal{R} t$.

(2) Let Δ and Θ be distributions over S and $\mathcal{R}$ be an equivalence relation. Then $\Delta \mathcal{R}^\dagger \Theta$ if and only if $\Delta(C) = \Theta(C)$ for all equivalence classes $C \in S/\mathcal{R}$, where $\Delta(C)$ stands for the accumulation probability $\sum_{s\in C} \Delta(s)$.

Given a binary relation $\mathcal{R} \subseteq S \times T$ and a set $S' \subseteq S$, we write $\mathcal{R}(S')$ for the set $\{t \in T \mid \exists s \in S' : s \mathcal{R} t\}$. A set S' is $\mathcal{R}$-closed if $\mathcal{R}(S') \subseteq S'$.

▶ Proposition 5. Let Δ and Θ be distributions over finite sets S and T, respectively.

(1) $\Delta\ \mathcal{R}^\dagger\ \Theta$ if and only if $\Delta(S') \leq \Theta(\mathcal{R}(S'))$ for all $S' \subseteq S$.
(2) If $\mathcal{R}$ is a preorder, then $\Delta\ \mathcal{R}^\dagger\ \Theta$ if and only if $\Delta(S') \leq \Theta(S')$ for each $\mathcal{R}$-closed set $S' \subseteq S$.

Besides the above interesting properties, the lifting operation has an intrinsic connection with some important concepts in mathematics, notably *the Kantorovich metric* [33]. For example, it turns out that our lifting of binary relations from states to distributions nicely corresponds to the lifting of metrics from states to distributions by using the Kantorovich metric. In addition, the lifting operation is closely related to *the maximum flow problem* in optimisation theory. This observation initially made by Baier *et al.* is crucial for designing decision algorithms for probabilistic bisimulations and simulations [1, 48].

2.3 Probabilistic Bisimulation

With a solid base of the lifting operation, we can proceed to define a probabilistic version of bisimulation. Let s and t be two states in a pLTS. We say t can simulate the behaviour of s if whenever the latter can exhibit some action, say a, and lead to distribution Δ then the former can also perform a and lead to a distribution, say Θ, which then in turn can mimic Δ in successor states. We are interested in defining a relation between two states, but it is expressed by invoking a relation between two distributions. To formalise the mimicking of one distribution by the other, we make use of the lifting operation investigated in Section 2.2.

▶ **Definition 6.** A relation $\mathcal{R} \subseteq S \times S$ is a *probabilistic simulation* if $s\ \mathcal{R}\ t$ implies
- if $s \xrightarrow{a} \Delta$ then there exists some Θ such that $t \xrightarrow{a} \Theta$ and $\Delta\ \mathcal{R}^\dagger\ \Theta$.

If both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are probabilistic simulations, then $\mathcal{R}$ is a *probabilistic bisimulation*. The largest probabilistic bisimulation, denoted by $\sim_s$, is called *(state-based) probabilistic bisimilarity*.

Let's look at the two pLTSs in Figure 1. It is easy to check that the top node in diagram (a) and that in diagram (b) are related by $\sim_s$.

Various characterisations of probabilistic bisimilarity by probabilistic versions of Hennessy-Milner logic [28] have appeared in the literature. In particular, if we confine ourselves to reactive pLTSs, then there are neat logical characterisations even without negation. For example, Desharnais et al. [14] uses a logic with the following grammar

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \langle a \rangle_q \varphi$$

where q is any rational number in the unit interval $[0, 1]$ and a ranges over the fixed set of labels of a given reactive pLTS. The formula $\top$ can always be satisfied. The formula $\varphi \wedge \varphi$ stands for the usual conjunction. The formula $\langle a \rangle_q \varphi$ is satisfied by state s if the probability that s can make an a-labelled transition to the set of states satisfying φ exceeds p. The characterisation result of [14] holds for reactive pLTSs with continuous state spaces. For reactive pLTSs with countable state spaces, a simpler proof of that result is given in [12]. Most other characterisations also employ modalities indexed with numbers. This fits in our intuition: if two states are not bisimilar, then they may satisfy a property with different probabilities, so by fiddling with the numbers we can construct a formula that can tell apart the two states. The only exception is the one given in [44], which shows that, for reactive probabilistic processes, probabilistic bisimilarity can be characterised by a surprisingly simple logic.

Let $\mathcal{L}$ be the set of formulae defined by the grammar

$$\phi\ ::=\ \top \mid \langle \phi, \phi \rangle \mid \langle a \rangle \phi$$

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where a ranges over the set of labels of a reactive pLTS. A state s satisfies a formula ϕ with certain probability, given by $Pr(s, \phi)$ defined as follows:

$$\begin{array}{rcl} Pr(s, \top) & = & 1 \\ Pr(s, \langle \phi_1, \phi_2 \rangle) & = & Pr(s, \phi_1) \cdot Pr(s, \phi_2) \\ Pr(s, \langle a \rangle \phi) & = & \begin{cases} \sum_{s' \in S} \Delta(s') \cdot Pr(s', \phi) & \text{if } s \xrightarrow{a} \Delta \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

We call $\langle \phi_1, \phi_2 \rangle$ a *conjunction* of two formulae ϕ_1 and ϕ_2, which models the copying capacity of probabilistic testing originally considered in [36]. Note that conjunction is given the arithmetic interpretation as multiplication, which differs from many other logical characterisations of probabilistic bisimilarity. The formula $\langle a \rangle \phi$ measures the probability that a state performs action a and then its successor states sastisfy ϕ.

The logic $\mathcal{L}$ induces a natural logical equivalence, written $=_{\mathcal{L}}$, by letting $s_1 =_{\mathcal{L}} s_2$ if $Pr(s_1, \phi) = Pr(s_2, \phi)$ for any $\phi \in \mathcal{L}$ and states s_1 and s_2. In [44] van Breugel et al. consider labelled Markov processes with continuous state spaces and they show that probabilistic bisimilarity coincides with the above notion of logical equivalence. Their proof involves advanced machinery such as the Lawson topology on probabilistic powerdomains [31] and Banach algebras. If we confine ourselves to finite-state reactive pLTSs, it is possible to avoid all the advanced machinery and give an elementary proof of the coincidence of $\sim_s$ with $=_{\mathcal{L}}$, as recently demonstrated in [7].

2.4 Distribution-Based Bisimulation

In Definition 6 we compare the behaviour of two states, and then resort to the lifting operation when talking about the simulation of one distribution by another. Alternatively, it is possible to consider subdistributions as first-class citizens and directly define a relation that compares subdistributions. In order to do so, we first define a transition relation between subdistributions.

▶ **Definition 7.** With a slight abuse of notation, we also use the notation $\xrightarrow{a}$ to stand for the transition relation between subdistributions, which is the smallest relation satisfying the following three rules:

(1) if $s \xrightarrow{a} \Delta$ then $\overline{s} \xrightarrow{a} \Delta$;

(2) if $s \not\xrightarrow{a}$ then $\overline{s} \xrightarrow{a} \varepsilon$;

(3) if $\Delta_i \xrightarrow{a} \Theta_i$ for all $i \in I$ then $(\sum_{i \in I} p_i \cdot \Delta_i) \xrightarrow{a} (\sum_{i \in I} p_i \cdot \Theta_i)$, where I is a finite index set and $\sum_{i \in I} p_i \leq 1$.

Note that if $\Delta \xrightarrow{a} \Delta'$ then some (not necessarily all) states in the support of Δ can perform action a. Those states that cannot enable action a contribute nothing for Δ'.

▶ **Definition 8.** Let $\sim_d \subseteq \mathcal{D}_{sub}(S) \times \mathcal{D}_{sub}(S)$ be the largest symmetric relation such that if $\Delta \sim_d \Theta$ then $|\Delta| = |\Theta|$ and $\Delta \xrightarrow{a} \Delta'$ implies the existence of some Θ' such that $\Theta \xrightarrow{a} \Theta'$ and $\Delta' \sim_d \Theta'$.

The distribution-based bisimilarity $\sim_d$ is shown in [6] as a sound and complete coinductive proof technique for linear contextual equivalence, a natural extensional behavioural equivalence for functional programs. In the literature there are several proposals of distribution-based bisimilarities [23, 25, 9, 17, 29], and some typical ones are compared in [16].

3 Quantum Bisimulation

In this section, we will see that quantum bisimulations can be obtained by extending probabilistic bisimulations in a smooth way.

As is well known, it is very difficult to guarantee the correctness of classical communication protocols at the design stage, and some simple protocols were eventually found to have fundamental flaws. One expects that the design of complex quantum protocols is at least as error-prone, if not more, than in the classical case. Bisimulation and its associated coinduction proof technique have also been explored in quantum concurrency theory.

Due to the presence of measurements, quantum processes exhibit probabilistic behaviour. It is then natural to define the operational semantics of a quantum process in turns of a pLTS, on which the probabilistic bisimulations we discussed before, with some modifications, may play a role in providing a coinduction proof technique for quantum processes. Note that in the quantum setting, bisimulations are defined to be relations over configurations that are pairs of a quantum process and a density operator describing the state of environment quantum systems. Below we illustrate this idea in the framework of a quantum process algebra.

3.1 Quantum Bisimulation for qCCS

We first briefly review the syntax and semantics of a quantum extension of value-passing CCS [37, 26], called qCCS, studied in [19, 47, 20, 21], and the definition of open bisimulation between qCCS processes presented in [5]; the idea can be applied in other quantum process algebras such as CQP [24] and QPAlg [32].

We assume three types of data in qCCS: Bool for booleans, real numbers Real for classical data, and qubits Qbt for quantum data. Let $cVar$, ranged over by $x, y, \ldots$, be the set of classical variables, and $qVar$, ranged over by $q, r, \ldots$, the set of quantum variables. It is assumed that $cVar$ and $qVar$ are both countably infinite. We assume a set Exp of classical data expressions over `Real`, which includes $cVar$ as a subset and is ranged over by $e, e', \ldots$, and a set of boolean-valued expressions $BExp$, ranged over by $b, b', \ldots$. We further assume that only classical variables can occur free in both data expressions and boolean expressions. Let $cChan$ be the set of classical channel names, ranged over by $c, d, \ldots$, and $qChan$ the set of quantum channel names, ranged over by $\mathsf{c}, \mathsf{d}, \ldots$. We often abbreviate a sequence of distinct variables $\{q_1, \ldots, q_n\}$ into $\widetilde{q}$.

Based on these notations, the syntax of qCCS terms can be given by the Backus-Naur form

$$U \quad ::= \quad \mathbf{nil} \mid K(\widetilde{e}, \widetilde{q}) \mid \alpha.U \mid U + U \mid U \| U \mid \mathbf{if}\ b\ \mathbf{then}\ U$$

$$\alpha \quad ::= \quad \tau \mid c?x \mid c!e \mid \mathsf{c}?q \mid \mathsf{c}!q \mid \mathcal{E}[\widetilde{q}] \mid M[\widetilde{q}; x]$$

where $c \in cChan$, $x \in cVar$, $\mathsf{c} \in qChan$, $q \in qVar$, $\widetilde{q} \subseteq qVar$, $e \in Exp$, $\widetilde{e} \subseteq Exp$, τ is the silent action, $b \in BExp$, $K(\widetilde{x}, \widetilde{q})$ is a process constant with a defining equation $K(\widetilde{x}, \widetilde{q}) \stackrel{def}{=} U$, and $\mathcal{E}$ and M are respectively a trace-preserving super-operator and a non-degenerate projective measurement applying on the Hilbert space associated with the systems $\widetilde{q}$. In this paper, we assume all super-operators are completely positive.

The notion of free classical variables in quantum processes, denoted by $fv(\cdot)$, can be defined in the usual way with the only modification that the quantum measurement prefix $M[\widetilde{q}; x]$ has binding power on x. A quantum process term U is closed if $fv(U) = \emptyset$. We let $\mathcal{U}$, ranged over by $U, V, \cdots$, be the set of all qCCS terms, and $\mathcal{P}$, ranged over by $P, Q, \cdots$, the set of closed terms.

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The process constructs we give here are quite similar to those in classical CCS, and they also have similar intuitive meanings: **nil** stands for a process which does not perform any action; $c?x$ and $c!e$ are respectively classical input and classical output, while $\mathsf{c}?q$ and $\mathsf{c}!q$ are their quantum counterparts. $\mathcal{E}[\widetilde{q}]$ denotes the action of performing the super-operator $\mathcal{E}$ on the qubits $\widetilde{q}$ while $M[\widetilde{q};x]$ measures the qubits $\widetilde{q}$ according to M and the measurement outcome is substituted for the classical variable x. The binary sum $+$ models nondeterministic choice: $U+V$ behaves like either U or V depending on the choice of the environment. $\|$ denotes the usual parallel composition. Finally, **if** b **then** U is the standard conditional choice where U can be executed only if b is $\mathtt{tt}$.

We now turn to the operational semantics of qCCS. For each quantum variable $q \in qVar$, we assume a 2-dimensional Hilbert space $\mathcal{H}_q$ to be the state space of the q-system. For any $S \subseteq qVar$, we denote $\mathcal{H}_S = \bigotimes_{q\in S}\mathcal{H}_q$. In particular, $\mathcal{H} = \mathcal{H}_{qVar}$ is the state space of the whole environment consisting of all the quantum variables. Note that $\mathcal{H}$ is a countably-infinite dimensional Hilbert space.

Suppose P is a closed quantum process. A pair of the form $\langle P,\rho\rangle$ is called a *configuration*, where $\rho \in \mathcal{D}(\mathcal{H})$ is a density operator on $\mathcal{H}$ (As $\mathcal{H}$ is infinite dimensional, ρ should be understood as a density operator on some finite dimensional subspace of $\mathcal{H}$ which contains $\mathcal{H}_{qv(P)}$). The set of configurations is denoted by Con, and ranged over by $\mathcal{C},\mathcal{D},\cdots$. Let

$$Act \quad = \quad \{\tau\} \cup \{c?v, c!v \mid c \in cChan, v \in \mathsf{Real}\} \cup \{\mathsf{c}?r, \mathsf{c}!r \mid \mathsf{c} \in qChan, r \in qVar\}.$$

Let $\mathcal{D}(Con)$, ranged over by $\Delta,\Theta,\cdots$, be the set of all finite-supported probabilistic distributions over Con. Then the operational semantics of qCCS can be given by the pLTS $\langle Con, Act, \longrightarrow\rangle$, where $\longrightarrow \subseteq Con \times Act \times \mathcal{D}(Con)$ is the smallest relation satisfying some inference rules. Here we select two rules related to super-operator application and quantum measurements; the others can be found in [5].

$$\textit{(Oper)}\quad \langle \mathcal{E}[\tilde{q}].P,\rho\rangle \xrightarrow{\tau} \langle P, \mathcal{E}_{\tilde{q}}(\rho)\rangle \qquad\qquad \textit{(Meas)}\quad \frac{M=\sum_{i\in I}\lambda_i E^i \qquad p_i = tr(E^i_{\tilde{q}}\rho)}{\langle M[\widetilde{q};x].P,\rho\rangle \xrightarrow{\tau} \sum_{i\in I} p_i\langle P[\lambda_i/x], E^i_{\widetilde{q}}\rho E^{i\dagger}_{\widetilde{q}}/p_i\rangle}$$

In rule *(Meas)*, $E^i_{\tilde{q}}$ denotes the operator E^i acting on the quantum systems $\tilde{q}$ and $tr(E^i_{\tilde{q}}\rho)$ stands for the trace of $E^i_{\tilde{q}}\rho$. This rule tells us that a measurement on the quantum system $\tilde{q}$ entails a probabilistic transition; each candidate configuration $\langle P[\lambda_i/x], E^i_{\tilde{q}}\rho E^{i\dagger}_{\tilde{q}}/p_i\rangle$ occurs with probability $tr(E^i_{\tilde{q}}\rho)$.

Let $\mathcal{C} = \langle P,\rho\rangle$. We use the notation $qv(\mathcal{C}) := qv(P)$ for free quantum variables and $\mathrm{env}(\mathcal{C}) := \mathrm{tr}_{qv(P)}(\rho)$ for partial traces. Let $\Delta = \sum_{i\in I} p_i \cdot \overline{\langle P_i,\rho_i\rangle}$. We write $\mathcal{E}(\Delta)$ for the distribution $\sum_{i\in I} p_i \cdot \overline{\langle P_i, \mathcal{E}(\rho_i)\rangle}$. In addition, we let $qv(\Delta) := \bigcup_{i\in I} qv(P_i)$ and $\mathrm{env}(\Delta) := \sum_{i\in I} p_i \cdot \mathrm{tr}_{qv(P_i)}(\rho_i)$.

▶ **Definition 9.** A symmetric relation $\mathcal{R} \subseteq Con \times Con$ is called an open bisimulation if for any $\mathcal{C},\mathcal{D} \in Con$, $\mathcal{C}\ \mathcal{R}\ \mathcal{D}$ implies that

(1) $qv(\mathcal{C}) = qv(\mathcal{D})$, and $\mathrm{env}(\mathcal{C}) = \mathrm{env}(\mathcal{D})$,

(2) for any trace-preserving super-operator $\mathcal{E}$ acting on $\mathcal{H}_{\overline{qv(\mathcal{C})}}$ (Again, $\mathcal{E}$ should be understood as a super-operator on some finite dimensional subspace of $\mathcal{H}_{\overline{qv(\mathcal{C})}}$), whenever $\mathcal{E}(\mathcal{C}) \xrightarrow{\alpha} \Delta$, there exists Θ such that $\mathcal{E}(\mathcal{D}) \xrightarrow{\alpha} \Theta$ and $\Delta\ \mathcal{R}^\dagger\ \Theta$.

Two quantum configurations $\mathcal{C}$ and $\mathcal{D}$ are open bisimilar, denoted by $\mathcal{C} \sim_o \mathcal{D}$, if there exists an open bisimulation $\mathcal{R}$ such that $\mathcal{C}\ \mathcal{R}\ \mathcal{D}$.

Here we are using exactly the same lifting operation as that in the probabilistic case (cf. Definition 2). The above definition is inspired by the work of Sangiorgi [42], where a

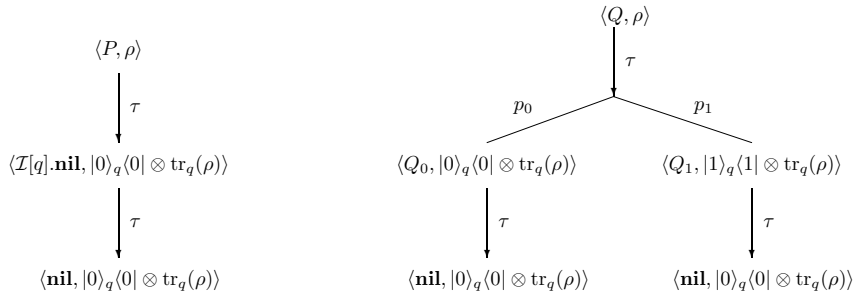

Figure 2 pLTSs for the two ways of setting a quantum system to $|0\rangle$

notion of bisimulation is defined for the π-calculus [38, 42] by treating name instantiation in an "open" style (name instantiation happens before any transition). Here we deal with super-operator application in an "open" style, but the instantiation of variables can be in an "early" style (variables are instantiated when input actions are performed). For example, the operational semantics given in [5] is essentially an early semantics.

To illustrate the operational semantics and open bisimulation presented in this section, we give a simple example.

▶ **Example 10.** This example shows two alternative ways of setting a quantum system to the pure state $|0\rangle$. Let $P \stackrel{def}{=} Set^0[q].\mathcal{I}[q].\mathbf{nil}$ and

$$Q \stackrel{def}{=} M_{0,1}[q; x].(\mathbf{if}\ x = 0\ \mathbf{then}\ \mathcal{I}[q].\mathbf{nil}\ +\mathbf{if}\ x = 1\ \mathbf{then}\ \mathcal{X}[q].\mathbf{nil}),$$

where $Set^0 = \{|0\rangle\langle 0|, |0\rangle\langle 1|\}$, $M_{0,1}$ is the 1-qubit measurement according to the computational basis $\{|0\rangle, |1\rangle\}$, $\mathcal{I}$ is the identity super-operator, and $\mathcal{X}$ is the Pauli-X super-operator. For any $\rho \in \mathcal{D}(\mathcal{H})$, the pLTSs rooted by $\langle P, \rho\rangle$ and $\langle Q, \rho\rangle$ respectively are depicted in Figure 2 where

$$Q_0 \stackrel{def}{=} \mathbf{if}\ 0 = 0\ \mathbf{then}\ \mathcal{I}[q].\mathbf{nil}\ +\mathbf{if}\ 0 = 1\ \mathbf{then}\ \mathcal{X}[q].\mathbf{nil},$$
$$Q_1 \stackrel{def}{=} \mathbf{if}\ 1 = 0\ \mathbf{then}\ \mathcal{I}[q].\mathbf{nil}\ +\mathbf{if}\ 1 = 1\ \mathbf{then}\ \mathcal{X}[q].\mathbf{nil},$$

and $p_i = \mathrm{tr}(|i\rangle\langle i|_q \cdot \rho)$. Note that both P and Q are free of quantum input. We can show $P \sim_o Q$ easily by verifying that the relation $\mathcal{R} \cup \mathcal{R}^{-1}$, where

$$\begin{aligned}\mathcal{R} = \{&(\langle P, \rho\rangle, \langle Q, \rho\rangle), (\langle \mathcal{I}[q].\mathbf{nil}, \rho_0\rangle, \langle Q_0, \rho_0\rangle),\\ &(\langle \mathcal{I}[q].\mathbf{nil}, \rho_0\rangle, \langle Q_1, \rho_1\rangle), (\langle \mathbf{nil}, \rho_0\rangle, \langle \mathbf{nil}, \rho_0\rangle) : \rho \in \mathcal{D}(\mathcal{H})\}\end{aligned}$$

and $\rho_i = |i\rangle\langle i|_q \otimes \mathrm{tr}_q\rho$, is an open bisimulation.

3.2 A Useful Proof Technique

In Definition 9 super-operator application and transitions are considered at the same time. In fact, we can separate the two issues and approach the concept of open bisimulation in an incremental way, which turns out to be very useful when proving that two configurations are bisimilar.

▶ **Definition 11.** A relation $\mathcal{R} \subseteq Con \times Con$ is closed under super-operator application if $\mathcal{C}\ \mathcal{R}\ \mathcal{D}$ implies $\mathcal{E}(\mathcal{C})\ \mathcal{R}\ \mathcal{E}(\mathcal{D})$ for any trace-preserving super-operator $\mathcal{E}$ acting on $\mathcal{H}_{\overline{qv(\mathcal{C})}}$.

▶ **Definition 12.** A relation $\mathcal{R} \subseteq Con \times Con$ is a *ground simulation* if $\mathcal{C}\ \mathcal{R}\ \mathcal{D}$ implies that $qv(\mathcal{C}) = qv(\mathcal{D})$, $\mathrm{env}(\mathcal{C}) = \mathrm{env}(\mathcal{D})$, and

- whenever $\mathcal{C} \xrightarrow{\alpha} \Delta$, there is some distribution Θ with $\mathcal{D} \xrightarrow{\alpha} \Theta$ and $\Delta\ \mathcal{R}^{\dagger}\ \Theta$.

A relation $\mathcal{R}$ is a *ground bisimulation* if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are ground simulations.

The following property is shown in [5].

▶ Proposition 13. $\sim_o$ is the largest ground bisimulation that is closed under all super-operator applications.

Proposition 13 provides us with a useful proof technique: in order to show that two configurations $\mathcal{C}$ and $\mathcal{D}$ are open bisimilar, it suffices to exhibit a binary relation including the pair $(\mathcal{C}, \mathcal{D})$, and then to check that the relation is a ground bisimulation and is closed under all super-operator application. This is analogous to a proof technique of open bisimulation for the π-calculus [42], where name instantiation is playing the same role as super-operator application here.

3.3 Distribution-Based Quantum Bisimulation

The distribution-based bisimulation defined in Section 2.4 can also be extended to the quantum setting.

▶ **Definition 14.** A relation $\mathcal{R} \subseteq \mathcal{D}(Con) \times \mathcal{D}(Con)$ is a *distribution-based ground simulation* if $\Delta\ \mathcal{R}\ \Theta$ implies that $qv(\Delta) = qv(\Theta)$, $\mathrm{env}(\Delta) = \mathrm{env}(\Theta)$, and

- whenever $\Delta \xrightarrow{\alpha} \Delta'$, there is some subdistribution Θ' with $\Theta \xrightarrow{\alpha} \Theta'$ and $\Delta'\ \mathcal{R}\ \Theta'$.

A relation $\mathcal{R}$ is a *distribution-based ground bisimulation* if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are distribution-based ground simulations.

A relation $\mathcal{R}$ is a distribution-based bisimulation if it is a distribution-based ground bisimulation, and is closed under super-operator applications.

Note that the distribution-based bisimulation given in Definition 14 is slightly coarser than that considered in [22], for the same reason as the comparison of the corresponding probabilistic bisimulations [16].

In quantum mechanics, a fundamental principle is the no-cloning theorem of quantum resources. From a type-theoretic point of view, quantum resources are linear and can be described by linear types in quantum programming languages. How to define appropriate program equivalences for this kind of languages is an interesting problem. In [8] a linear contextual equivalence is introduced to compare the behaviour of quantum programs. Two notions of bisimilarity, a state-based and a distribution-based are introduced as proof techniques for reasoning about higher-order quantum programs. Both notions of bisimilarity are sound with respect to the linear contextual equivalence, but only the distribution-based one turns out to be complete.

3.4 Symbolic Bisimulations

The quantum bisimulations introduced so far, either state-based or distribution-based, are generalised from the corresponding probabilistic bisimulations naturally and smoothly. A major problem with them is that they all resort to the instantiation of quantum variables by quantum states. As a result, to check whether or not two processes are bisimilar, we have to accompany them with arbitrarily chosen quantum states, and check if the resultant configurations are bisimilar. Note that all quantum states constitute a continuum. Therefore,

it seems that the verification of quantum bisimulations is infeasible from an algorithmic point of view.

Recall that for classical process algebras, Hennessy and Lin [27] introduced a notion of symbolic bisimulation to deal with possibly infinite classical data sets. As a quantum extension of value-passing CCS, the quantum process algebra qCCS has both (possibly infinite) classical data domain and (doomed-to-be infinite) quantum data domain. To overcome the additional difficulty caused by the infinity of all quantum states, we can make use of super-operator valued distributions, which allow us to fold the operational semantics of qCCS into a symbolic version and thus provide us with a notion of symbolic bisimulation. To check the symbolic bisimilarity of two quantum processes, only a finite number of process-superoperator pairs need to be considered, without appealing to quantum states. This idea has been successful in developing an algorithm to check the state-based ground bisimulation for quantum processes [18]. It would be interesting to purse this line of research so as to develop algorithms of checking the symbolic versions of other quantum bisimulations.

4 Concluding Remarks

We have briefly reviewed a few ingredients for formulating reasonable notions of probabilistic and quantum bisimulations.

(1) The lifting operation is the key of defining state-based probabilistic and quantum bisimualtions. It is mathematically interesting in itself because of the close connection with the Kantorovich metric and the maximum network flow problem.

(2) Distribution-based bisimulation is more relevant to quantum processes because it offers a coinductive proof technique for linear contextual equivalene, and linear resources are prominent in quantum computation.

(3) The symbolic approach is promising to yield feasible algorithms of checking quantum bisimulations.

There is a huge amount of literature on probabilistic bisimulations, and the current paper is by no means a complete survey. A more detailed account of probabilistic bisimulations is given in [4, Chapter3]. For quantum processes, a branching bisimulation is firstly proposed in [35]. However, it is not a congruence because it is not preserved by parallel composition. Quantum bisimulations that are congruence relations are given in [20, 18] and independently in [3]. Both of them are defined for concrete quantum transition systems, and are difficult to check with algorithms, which motivated the introduction of symbolic bisimulations for quantum processes [18].

In [34] a semi-automated tool is developed to verify security proofs based on a weak bisimulation similar to that given in Definition 9 for a finite fragment of qCCS. In that tool, security parameters and quantum states are represented as symbols, and some user-defined equations are used as rewriting rules for simplification. This differs from the symbolic semantics discussed in Section 3.4 as the latter is more in line with the idea investigated in [27] for value-passing CCS.

In the future, we believe that distribution-based symbolic bisimulations would be promising to be used in software tools in support of verifying quantum communication protocols. Some efforts are made in [22], which considers distribution-based bisimulations and the proofs are manual when reasoning about the behavioural equivalence of quantum processes. In order to deal with advanced protocols such as the quantum key distribution protocol BB84 [2], it would be helpful to have some tool support, for which symbolic semantics could play a role.

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